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On the explicit structure of $K_2(\mathbb{F}_pG)$ for G a finite abelian p-group

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ABSTRACT

Let $\mathbb{F}_p$ be a finite field with p a prime number and G a finite abelian p-group. We give the explicit structure of $K_2(\mathbb{F}_pG)$; in particular $K_2(\mathbb{F}_pG)$ is not an elementary abelian p-group when the p^2-rank of G is greater than 1.

1. Introduction

Let $\mathbb{F}$ be a finite field of characteristic p and G a finite abelian group; the p^i-rank of G is defined to be $\dim_{\mathbb{F}_p} G^{p^{i-1}}/G^{p^i}$. For elementary abelian p-groups G, Dennis et al. [1] calculated $K_2(\mathbb{F}G)$ and then got lower bounds for the order of $K_2(\mathbb{Z}G)$ and $Wh_2(G)$. For $p = 2$ and with the 4-rank of $G \leq 1$, Magurn [3] proved that $K_2(\mathbb{F}G)$ is an elementary abelian 2-group and gave Steinberg symbols as generators. We [2] extended Magurn's results to odd prime numbers p and calculated $K_2(\mathbb{F}G)$ for when the p^2-rank of $G \leq 1$. Now a natural question arises:

Question. If the p^2-rank of $G \geq 2$, what is the structure of $K_2(\mathbb{F}G)$?

If G is a finite abelian group and C_{p^r} a cyclic group of order p^r, by (3.1) in [2] we have the following decomposition formula:

$$K_2(\mathbb{F}[G \times C_{p^r}]) \cong K_2(\mathbb{F}G) \oplus K_2(\mathbb{F}G[t]/(t^{p^r}), (t)). \tag{1.1}$$

For G a finite abelian p-group, the order of the finite p-group $K_2(\mathbb{F}G)$ was given by Oliver [4], so the order of the finite p-group $K_2(\mathbb{F}H[t]/(t^{p^r}), (t))$ for any finite abelian p-group H can be calculated by using the above decomposition formula. After finding a sufficient number of Dennis–Stein symbols for generating $K_2(\mathbb{F}H[t]/(t^{p^r}), (t))$, we get the explicit structure of $K_2(\mathbb{F}H[t]/(t^{p^r}), (t))$. And by repeated use of (1.1), the explicit structure of $K_2(\mathbb{F}G)$ for arbitrary finite abelian p-group G is given.

The main results of this paper are the following two theorems.

Theorem 1.1. *Let $\mathbb{F}_p$ be a finite field with p a prime number. Let $G = C_{p^{\alpha_1}} \times \cdots \times C_{p^{\alpha_n}} = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_n \rangle$ be a finite abelian p-group and $\alpha_1, \ldots, \alpha_n \geq m$; then*

$$K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t)) \cong C_{p^m}^{|G^{p^m}|(np^{n+1}-(n-1)p^n-1)} \bigoplus_{i=1}^{m-1} C_{p^i}^{|G^{p^i}|(np^{m-n-i-1}(p^{n+1}-1)^2-(n-1)p^{-n}(p^n-1)^2)}.$$

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Theorem 1.2. *Let $\mathbb{F}_p$ be a finite field with p prime and G a finite abelian p-group of exponent p^e. Let r_i denote the p^i-rank of G. Then*

$$K_2(\mathbb{F}_pG) = C_{p^e}^{(r_e-1)(|G^{p^{e-1}}|-1)} \bigoplus_{i=1}^{e-1} C_{p^i}^{(r_i-1)(|G^{p^{i-1}}|-|G^{p^i}|)-(r_{i+1}-1)(|G^{p^i}|-|G^{p^{i+1}}|)}.$$

Theorem 1.2 extends the corresponding results in [1–3], showing that $K_2(\mathbb{F}_pG)$ is no longer an elementary abelian p-group when the p^2-rank of G is greater than 1.

2. Preliminaries

Let R be a commutative ring with unit and $I \subseteq \mathrm{rad}(R)$. By (1.4) of [3], the relative K_2-group $K_2(R, I)$ is generated by Dennis–Stein symbols $\langle a, b\rangle$ with a or b in I, satisfying the following relations:

$$\begin{array}{lll} (DS1) & \langle a,b\rangle = -\langle b,a\rangle & \text{if } a \in I; \\ (DS2) & \langle a,b\rangle + \langle a,c\rangle = \langle a, b+c-abc\rangle & \text{if } a \in I \text{ or } b,c \in I; \\ (DS3) & \langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle & \text{if } a \in I. \end{array}$$

The relations (2.1) and (2.2) in the following lemma are frequently used in the computation of Dennis–Stein symbols in this paper.

Lemma 2.1 (*See [5, Lemma 1.5]*). *Let R and I be as above. If R is an $\mathbb{F}_p$-algebra, then*

$$(2.1) \quad p^r\langle a,b\rangle = \langle a^{p^r} b^{p^r-1}, b\rangle.$$

For an arbitrary ring R and positive integer m,

$$(2.2) \quad \langle a, b^m\rangle = m\langle ab^{m-1}, b\rangle.$$

Let K be an unramified extension of the p-adic field $\mathbb{Q}_p$ with $[K : \mathbb{Q}_p] = f$ and A the valuation ring of K; then $A/(p)$ is just the finite field F_q with $q = p^f$, so Proposition 6.3 and Theorem 6.6 in [4] give the precise order of $K_2(\mathbb{F}_qG)$, which is essential for determining the structure of $K_2(\mathbb{F}_qG)$ through direct calculation of symbols.

Theorem 2.2 (*See [4, Proposition 6.3]*). *Let F_q be a finite field with $q = p^f$ and G a finite abelian p-group. If $\exp(G) = p^e$ and $r_i = p^i$-$rk(G)(1 \le i \le e)$, then*

$$ord_p|K_2(F_qG)| = f((r_1-1)|G| - (r_1-r_2)|G^p| - \cdots - (r_{e-1}-r_e)|G^{p^{e-1}}| - (r_e-1)).$$

3. The main results

Let R be a commutative ring and I a radical ideal of R. If $a, b \in I$ or $c \in I$, then by (DS2) we have the following equations in $K_2(R, I)$:

$$\langle a+b, c\rangle = \langle a,c\rangle + \langle b,c\rangle + \langle (1-(a+b-abc)c)^{-1}abc, c\rangle. \tag{3.1}$$

Let $b, c \in I$ or $a \in I$. As (3.1) we have

$$\langle a, b+c\rangle = \langle a,b\rangle + \langle a,c\rangle + \langle a, (1-(b+c-abc)a)^{-1}abc\rangle. \tag{3.2}$$

We first need three lemmas to prove Theorem 3.4 below, which gives a relatively small number of generators of $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$.

Lemma 3.1. *Let $a_l, \ldots, a_{p^m-1}$ be arbitrary elements of R, $l > 1$; then in $K_2(R[t]/(t^{p^m}), (t))$, $\langle a_lt^l + \cdots + a_{p^m-1}t^{p^m-1}, t\rangle$ is a sum of elements in $\{\langle at^i, t\rangle | l \le i \le p^m-1, a \in R\}$.*

Proof. If $l = p^m - 1$, the lemma is obviously true. Let $l = k$; by (3.1) we have

$$\langle a_kt^k + \cdots + a_{p^m-1}t^{p^m-1}, t\rangle = \langle a_kt^k, t\rangle + \langle a_{k+1}t^{k+1} + \cdots + a_{p^m-1}t^{p^m-1}, t\rangle + \langle \theta, t\rangle,$$

where $\theta \in (t^{2k+2})$. Since $k+1, 2k+2 > k$, by the induction hypothesis the lemma is true for $l = k$ and the lemma is proved. □

Lemma 3.2. *Let $a_l, b_1, \ldots, b_{p^m-1}$ be arbitrary elements of R, $1 \le l \le p^m-1$; then in $K_2(R[t]/(t^{p^m}), (t))$, $\langle a_lt^l, b_1t + \cdots + b_{p^m-1}t^{p^m-1}\rangle$ is a sum of elements of $\{\langle at^i, t\rangle, \langle at^j, b\rangle | i \ge l, j > l, a, b \in R\}$.*

Proof. By (3.2) we have

$$\langle a_lt^l, b_1t + \cdots + b_{p^m-1}t^{p^m-1}\rangle = \langle a_lt^l, b_1t\rangle + \langle a_lt^l, b_2t^2 + \cdots + b_{p^m-1}t^{p^m-1}\rangle + \langle a_lt^l, \theta\rangle,$$

where $\theta \in (t^{l+3})$ and by (DS3)

$$\langle a_l t^l, b_1 t\rangle = \langle a_l b_1 t^l, t\rangle + \langle a_l t^{l+1}, b_1\rangle.$$

Now an easy induction yields the lemma. □

Lemma 3.3. *Let $a_l, \dots, a_{p^m-1}$ and b be arbitrary elements of R; then $\langle a_l t^l + \cdots + a_{p^m-1}t^{p^m-1}, b\rangle$ is a sum of elements of $\{\langle at^i, b\rangle | a \in R, i \geq l\}$ in $K_2(R[t]/(t^{p^m}), (t))$.*

Proof. By (3.1) we have

$$\langle a_l t^l + \cdots + a_{p^m-1}t^{p^m-1}, b\rangle = \langle a_l t^l, b\rangle + \langle a_{l+1}t^{l+1} + \cdots + a_{p^m-1}t^{p^m-1}, b\rangle + \langle \theta, b\rangle,$$

where $\theta \in (t^{2l+1})$; now an easy induction yields the lemma. □

Theorem 3.4. *Let $\mathbb{F}_p$ be a finite field with p elements and $G = \langle\sigma_1\rangle \times \cdots \times \langle\sigma_n\rangle$ be a finite abelian p-group. Then $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$ can be generated by*

$$S = \{\langle gt^k, t\rangle, \langle gt^k, \sigma_i\rangle | g \in G, 1 \leq k < p^m, 1 \leq i \leq n\}.$$

Proof. By Proposition 1.7 in [5], $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$ is generated by elements $\langle at^i, t\rangle$ and $\langle at^i, b\rangle$ with $a, b \in \mathbb{F}_pG$ and $1 \leq i \leq p^m - 1$, and we define a filtration on $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$ using these elements. Let $F_0 = 0$ and:

(1) when $1 \leq k \leq p^m - 1$,
F_k = the subgroup generated by F_{k-1} and symbols of the type $\langle at^{p^m-k}, t\rangle$;
(2) when $p^m \leq k \leq 2p^m - 2$,
F_k = the subgroup generated by F_{k-1} and symbols of the type $\langle at^{2p^m-k-1}, b\rangle$.

Then $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{p^m-1} \subseteq F_{p^m} \subseteq \cdots \subseteq F_{2p^m-2} = K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$. To prove the theorem, it suffices to prove that the image of $S \cap F_k$ under the natural map $F_k \to F_k/F_{k-1}$ is a set of generators of F_k/F_{k-1}.

① For $1 \leq k \leq p^m - 1$, F_k/F_{k-1} is generated by $\overline{\langle at^{p^m-k}, t\rangle}$ with $a \in \mathbb{F}_pG$. Let a_1, a_2 be arbitrary elements of $\mathbb{F}_pG$; then by (3.1)

$$\langle a_1 t^{p^m-k}, t\rangle + \langle a_2 t^{p^m-k}, t\rangle = \langle (a_1 + a_2)t^{p^m-k}, t\rangle - \langle \theta, t\rangle,$$

where $\theta \in (t^{2p^m-2k+1})$ and $2p^m - 2k + 1 > p^m - k$. By Lemma 3.1, $\langle\theta, t\rangle \in F_{k-1}$. Since each element a of $\mathbb{F}_pG$ is an $\mathbb{F}_p$-linear combination of $g, g \in G$, the image of $\{\langle gt^{p^m-k}, t\rangle | g \in G\}$ generates F_k/F_{k-1} for $1 \leq k \leq p^m - 1$.

② For $p^m \leq k \leq 2p^m - 2$, F_k/F_{k-1} is generated by $\overline{\langle at^{2p^m-k-1}, b\rangle}$ with $a, b \in \mathbb{F}_pG$. Let b_1,b_2 be arbitrary elements of $\mathbb{F}_pG$; by (3.2),

$$\langle at^{2p^m-k-1}, b_1\rangle + \langle at^{2p^m-k-1}, b_2\rangle = \langle at^{2p^m-k-1}, b_1 + b_2\rangle - \langle at^{2p^m-k-1}, \theta\rangle,$$

where $\theta \in (t^{2p^m-k-1})$. By Lemma 3.2, we have $\langle at^{2p^m-k-1}, \theta\rangle \in F_{k-1}$. Then $\overline{\langle at^{2p^m-k-1}, b\rangle}$ can be generated by $\overline{\langle at^{2p^m-k-1}, \sigma_1^{h_1} \cdots \sigma_n^{h_n}\rangle}$. By (DS3) for $a' \in I$,

$$\langle a', \sigma_1^{h_1} \cdots \sigma_n^{h_n}\rangle = \sum_{i=1}^{n} h_i \langle a'\sigma_1^{h_1} \cdots \sigma_i^{h_i-1} \cdots \sigma_n^{h_n}, \sigma_i\rangle.$$

So $\overline{\langle at^{2p^m-k-1}, b\rangle}$ is generated by symbols in $\{\overline{\langle at^{2p^m-k-1}, \sigma_i\rangle} | a \in \mathbb{F}_pG, 1 \leq i \leq n\}$. Let a_1,a_2 be arbitrary elements of $\mathbb{F}_pG$; by (3.1),

$$\langle a_1 t^{2p^m-k-1}, \sigma_i\rangle + \langle a_2 t^{2p^m-k-1}, \sigma_i\rangle = \langle (a_1 + a_2)t^{2p^m-k-1}, \sigma_i\rangle + \langle \theta, \sigma_i\rangle,$$

where $\theta \in (t^{2(2p^m-k-1)})$. By Lemma 3.3, $\langle\theta, \sigma_i\rangle \in F_{k-1}$; then $\overline{\langle at^{2p^m-k-1}, \sigma_i\rangle}$ is generated by symbols $\{\overline{\langle gt^{2p^m-k-1}, \sigma_i\rangle} | g \in G\}$. Now the theorem is proved. □

The next lemma further reduces the number of Dennis–Stein symbols needed to generate $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$.

Lemma 3.5. *Let $\mathbb{F}_p$, G and S be as in Theorem 3.4. Let*

$$T_1 = \{\langle g^p t^k, t\rangle | g \in G, 1 \leq k < p^m\};$$

$$T_2 = \{\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i} g^p t^k, \sigma_i\rangle | 0 \leq l_1, \dots, l_{i-1} \leq p - 1, 0 \leq l_i \leq p - 2, g \in G, 2 \leq k < p^m, 1 \leq i \leq n\};$$

$$T_3 = \{\langle \sigma_i^{p-1} g^p t^k, \sigma_i\rangle | 1 \leq i \leq n, g \in G, 2 \leq k < p^m, k \equiv 0 \bmod p\}.$$

Set $T = T_1 \cup T_2 \cup T_3$. Then $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$ can be generated by Dennis–Stein symbols in $S \setminus T$.

Proof. By Theorem 3.4, we need only to show that each symbol in T is a sum of symbols in $S \setminus T$. Now we consider the symbols in T_1, T_2 and T_3 separately.

① Symbols in T_1.

Let $g = \sigma_1^{l_1} \cdots \sigma_n^{l_n}$. If $k+1 \not\equiv 0 \mod p$,

$$\begin{aligned}\langle g^p t^k, t\rangle &= \langle g^p, t^{k+1}\rangle + \langle t^k, g^p t\rangle \\ &= -\langle t^{k+1}, g^p\rangle - \langle g^p t, t^k\rangle \\ &= -p\langle g^{p-1}t^{k+1}, g\rangle - k\langle g^p t^k, t\rangle,\end{aligned}$$

and by (2.2) we have

$$(1+k)\langle g^p t^k, t\rangle = -p\langle g^{p-1}t^{k+1}, g\rangle = -p\sum_{i=1}^{n} l_i\langle g^p\sigma_i^{-1}t^{k+1}, \sigma_i\rangle.$$

Since $k+1 \not\equiv 0 \mod p$, $\langle g^p\sigma_i^{-1}t^{k+1}, \sigma_i\rangle \in S \setminus T$. Since $K_2(\mathbb{F}_p G[t]/(t^{p^m}), (t))$ is a finite abelian p-group, $\langle g^p t^k, t\rangle$ is a sum of symbols in $S \setminus T$ when $k+1 \not\equiv 0 \mod p$.

If $k+1 \equiv 0 \mod p$, let $k = lp-1$, $1 \le l \le p^{m-1}$; then $\langle g^p t^k, t\rangle = \langle g^p t^{lp-1}, t\rangle$. If $l = 1$,

$$\langle g^p t^{p-1}, t\rangle = p\langle g, t\rangle = -p\langle t, g\rangle = -p\sum_{i=1}^{n} l_i\langle g\sigma_i^{-1}t, \sigma_i\rangle.$$

Obviously $\langle g\sigma_i^{-1}t, \sigma_i\rangle \in S \setminus T$, and $\langle g^p t^{p-1}, t\rangle$ is a sum of symbols in $S \setminus T$.

If $l > 1$ and $g \notin G^p$,

$$\langle g^p t^{lp-1}, t\rangle = p\langle gt^{l-1}, t\rangle.$$

Also $\langle gt^{l-1}, t\rangle \in S \setminus T$.

If $l > 1$ and $g \in G^p$, let $g = g'^p$ for some $g' \in G$; then

$$\langle g^p t^{lp-1}, t\rangle = p\langle g'^p t^{l-1}, t\rangle. \tag{3.3}$$

Since $\langle g'^p t^{l-1}, t\rangle \in T_1$, we can repeat the discussion above to show that either $\langle g'^p t^{l-1}, t\rangle$ is a sum of symbols in $S \setminus T$ or $\langle g'^p t^{l-1}, t\rangle = p\langle g''^p t^{l'-1}, t\rangle$ by (3.3) when $g' \in G^p$ and $l = bp$, $b > 1$. But $l'-1 < l-1 < lp-1$, so after a finite number of steps we can show that $\langle g^p t^{lp-1}, t\rangle$ is a sum of symbols in $S \setminus T$ when $g \in G^p$ and $l > 1$.

② Symbols in T_2.

Let $g = \sigma_1^{l'_1} \cdots \sigma_n^{l'_n}$; then

$$\begin{aligned}\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p t^k, \sigma_i\rangle &= -\langle \sigma_i, \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p t^k\rangle \\ &= -\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}g^p, t^k\rangle - \langle \sigma_i t^k, \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p\rangle \\ &= -k\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}g^p t^{k-1}, t\rangle - p\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}g^{p-1}t^k, g\rangle - \langle \sigma_i g^p t^k, \sigma_1^{l_1}\cdots\sigma_i^{l_i}\rangle.\end{aligned}$$

For the last two terms in the above equation,

$$\begin{aligned}\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}g^{p-1}t^k, g\rangle &= \sum_{j<i} l'_j\langle \sigma_1^{l_1}\cdots\sigma_j^{l_j-1}\cdots\sigma_i^{l_i+1}g^p t^k, \sigma_j\rangle + l'_i\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p t^k, \sigma_i\rangle \\ &\quad + \sum_{j>i} l'_j\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}\sigma_j^{-1}g^p t^k, \sigma_j\rangle, \\ \langle \sigma_i g^p t^k, \sigma_1^{l_1}\cdots\sigma_i^{l_i}\rangle &= \sum_{j<i} l_j\langle \sigma_1^{l_1}\cdots\sigma_j^{l_j-1}\cdots\sigma_i^{l_i+1}g^p t^k, \sigma_j\rangle + l_i\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p t^k, \sigma_i\rangle,\end{aligned}$$

so we have

$$\begin{aligned}(1+pl'_i+l_i)\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p t^k, \sigma_i\rangle &= -k\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}g^p t^{k-1}, t\rangle - \sum_{j<i} l_j\langle \sigma_1^{l_1}\cdots\sigma_j^{l_j-1}\cdots\sigma_i^{l_i+1}g^p t^k, \sigma_j\rangle \\ &\quad - p\left(\sum_{j<i} l'_j\langle \sigma_1^{l_1}\cdots\sigma_j^{l_j-1}\cdots\sigma_i^{l_i+1}g^p t^k, \sigma_j\rangle + \sum_{j>i} l'_j\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}\sigma_j^{-1}g^p t^k, \sigma_j\rangle\right).\end{aligned}$$

Since $1 \le l_i + 1 \le p-1$, and so $\sigma_i^{l_i+1} \notin G^p$, $\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}g^p t^{k-1}, t\rangle$, $\langle \sigma_1^{l_1}\cdots\sigma_j^{l_j-1}\cdots\sigma_i^{l_i+1}g^p t^k, \sigma_j\rangle (j < i)$ and $\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i+1}\sigma_j^{-1}g^p t^k, \sigma_j\rangle (j > i)$ in the above equation are in $S \setminus T$. And because $(1+pl'_i+l_i) \not\equiv 0 \mod p$, $K_2(\mathbb{F}_p G[t]/(t^{p^m}), (t))$ is a finite abelian p-group, so $\langle \sigma_1^{l_1}\cdots\sigma_i^{l_i}g^p t^k, \sigma_i\rangle$ in T_2 is a sum of symbols in $S \setminus T$.

③Symbols in T_3.

Let $k = lp$, $1 \le l \le p^{m-1} - 1$; then

$$\langle \sigma_i^{p-1}g^p t^{lp}, \sigma_i\rangle = p\langle gt^l, \sigma_i\rangle.$$

If $\langle gt^l, \sigma_i\rangle \in T_2$, by the discussion in ②, $\langle gt^l, \sigma_i\rangle$ is a sum of symbols in $S \setminus T$; if $\langle gt^l, \sigma_i\rangle \in T_3$, then $\langle gt^l, \sigma_i\rangle = p\langle g't^{l'}, \sigma_i\rangle$ and $l' < l < lp$. But if $l = 1$, $\langle gt^l, \sigma_i\rangle \in S \setminus T$. So after a finite number of steps, we can show that $\langle \sigma_i^{p-1} g^p t^{lp}, \sigma_i\rangle$ is a sum of Dennis–Stein symbols in $S \setminus T$. Now the lemma is proved. □

Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $\mathbb{N}$ denote nonnegative integers and

$$V = \{(l_1, \dots, l_n) \in \mathbb{N}^n | 0 \le l_j < p^{\alpha_j},\ 1 \le j \le n\}.$$

For each k, $1 \le k < p^m$, we shall define a partition $N_{k,1}, \dots, N_{k,m-s}$ of V. We shall show below that if $(l_1, \dots, l_n) \in N_{k,m-s}$, the exponent of $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle$ is $\le p^{m-s}$; if $(l_1, \dots, l_n) \in N_{k,i}$, $1 \le i < m - s$, the exponent of $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle - \langle \sigma_1^{l'_1} \cdots \sigma_n^{l'_n} t^k, x\rangle$ is $\le p^i$. The meanings of s, x and $(l'_1, \dots, l'_n)$ are explained below.

If $p^s \le k < p^{s+1}$, $0 \le s \le m-1$, set

$$N'_{k,i} = \{(l_1, \dots, l_n) \in \mathbb{N}^n | 0 \le l_j < p^{\alpha_j - i+1},\ 1 \le j \le n\}, \quad 1 \le i \le m - s.$$

Then $N'_{k,m-s} \subseteq \cdots \subseteq N'_{k,1}$, and let $N_{k,m-s} = N'_{k,m-s}$, $N_{k,i} = N'_{k,i} \setminus N'_{k,i+1}$, $1 \le i < m - s$; so $N_{k,1}, \dots, N_{k,m-s}$ is a partition of V. If $i < m - s$ and $(l_1, \dots, l_n) \in N_{k,i}$, then there exists only one $(l'_1, \dots, l'_n) \in N'_{k,i+1}$ such that $l_j \equiv l'_j \mod p^{\alpha_j - i}$.

For convenience, let x stand for t or σ_j, $1 \le j \le n$. If $(l_1, \dots, l_n) \in N_{k,m-s}$, since $k \cdot p^{m-s} \ge p^m$,

$$p^{m-s}\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle = \langle (\sigma_1^{l_1} \cdots \sigma_n^{l_n})^{p^{m-s}} x^{p^{m-s}-1} t^{k\cdot p^{m-s}}, x\rangle = 0,$$

so the order of $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle$ is $\le p^{m-s}$. If $(l_1, \dots, l_n) \in N_{k,i}$, $i < m - s$,

$$p^i\big(\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle - \langle \sigma_1^{l'_1} \cdots \sigma_n^{l'_n} t^k, x\rangle\big) = 0,$$

so the order of $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle - \langle \sigma_1^{l'_1} \cdots \sigma_n^{l'_n} t^k, x\rangle \le p^i$. For $i < m-s$ and $(l_1, \dots, l_n) \in N_{k,i}$, if $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle \in S \setminus T$, then $\langle \sigma_1^{l'_1} \cdots \sigma_n^{l'_n} t^k, x\rangle \in S \setminus T$, so if we replace $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle \in S \setminus T$ by $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle - \langle \sigma_1^{l'_1} \cdots \sigma_n^{l'_n} t^k, x\rangle$, the new set is still a generating set of $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$ and the number of these generators does not change. We still use $S \setminus T$ to denote it.

Now we begin to count the number of Dennis–Stein symbols in the new generating set $S \setminus T$. For $p^s \le k < p^{s+1}$, $0 \le s \le m-1$, let $l_{k,x}^{m-s}$ denote the number of $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle \in S \setminus T$ with $(l_1, \dots, l_n) \in N_{k,m-s}$. Let $l_{k,x}^i$ denote the number of $\langle \sigma_1^{l_1} \cdots \sigma_n^{l_n} t^k, x\rangle - \langle \sigma_1^{l'_1} \cdots \sigma_n^{l'_n} t^k, x\rangle \in S \setminus T$ with $(l_1, \dots, l_n) \in N_{k,i}$, $i < m - s$. Using the definitions of T_i in Lemma 3.5 we can get the value of $l_{k,x}^i$. If $x = t$ and $p^s \le k < p^{s+1}$,

$$l_{k,t}^{m-s} = |G|p^{-n(m-s)}(p^n - 1), \qquad l_{k,t}^i = |G|p^{-ni}(p^n - 1)(1 - p^{-n}), \quad i < m - s.$$

Now let $x = \sigma_j$, $1 \le j \le n$. If $k = 1$,

$$l_{k,\sigma_j}^m = |G|p^{-nm}p^n, \qquad l_{k,\sigma_j}^i = |G|p^{-ni}p^n(1 - p^{-n}), \quad i < m.$$

When $p^s \le k < p^{s+1}$, $2 \le k$ and $k \not\equiv 0 \mod p$,

$$l_{k,\sigma_j}^{m-s} = |G|p^{-n(m-s)}(p^n - p^j + p^{j-1});$$

$$l_{k,\sigma_j}^i = |G|p^{-ni}(p^n - p^j + p^{j-1})(1 - p^{-n}), \quad 1 \le i < m - s;$$

When $p^s \le k < p^{s+1}$, $2 \le k$ and $k \equiv 0 \bmod p$,

$$l_{k,\sigma_j}^{m-s} = |G|p^{-n(m-s)}(p^n - p^j + p^{j-1} - 1);$$

$$l_{k,\sigma_j}^i = |G|p^{-ni}(p^n - p^j + p^{j-1} - 1)(1 - p^{-n}), \quad 1 \le i < m - s.$$

Let

$$\beta_m = \sum_{k=1}^{p-1}\left(l_{k,t}^m + \sum_{j=1}^n l_{k,\sigma_j}^m\right) = |G|p^{-mn}(np^{n+1} - (n-1)p^n - 1),$$

$$\beta_i = \sum_{k=1}^{p^{m-i+1}-1}\left(l_{k,t}^i + \sum_{j=1}^n l_{k,\sigma_j}^i\right) = |G|p^{-ni}(np^{m-n-i-1}(p^{n+1} - 1)^2 - (n-1)p^{-n}(p^n - 1)^2).$$

So the order of $K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t)) \le \prod_{i=1}^m p^{i\beta_i} = p^{|G|(np^m - p^{-mn} - (n-1))}$. Since $K_2(\mathbb{F}_p[G \times C_{p^m}]) \cong K_2(\mathbb{F}_pG) \oplus K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))$, by Theorem 2.2, we know that

$$|K_2(\mathbb{F}_pG[t]/(t^{p^m}), (t))| = |K_2(\mathbb{F}_p[G \times C_{p^m}])|/|K_2(\mathbb{F}_pG)| = p^{|G|(np^m - p^{-mn} - (n-1))}.$$

So

$$K_2(\mathbb{F}_pG[t]/(t^{p^m}),(t)) = \bigoplus_{i=1}^{m} C_{p^i}^{\beta_i},$$

and the theorem is proved. □

Proof of Theorem 1.2. We use induction on the p-rank of G. As noted in [3], $K_2(\mathbb{F}_pG) = 1$ if G is finite cyclic. Let C_{p^l} be a direct summand of least order in the decomposition of G as a direct sum of cyclic p-groups and $H \leq G$ such that $G = H \times C_{p^l}$, $l \leq e$; so if G is not cyclic, G and H have the same exponent p^e. Let r_i' and r_i denote the p^i-ranks of G and H respectively. By (3.1) in [2],

$$K_2(\mathbb{F}_pG) \cong K_2(\mathbb{F}_pH) \oplus K_2(\mathbb{F}_pH[t]/(t^{p^l}),(t)).$$

Let

$$K_2(\mathbb{F}_pH) = \bigoplus_{i=1}^{e} C_{p^i}^{\alpha_i}, \qquad K_2(\mathbb{F}_pH[t]/(t^{p^l}),(t)) = \bigoplus_{i=1}^{l} C_{p^i}^{\beta_i}.$$

By the induction hypothesis, all the α_i are known.

We will consider $e > l$ and $e = l$ separately. First suppose that $e > l$. For $l < i \leq e$, since $r_i = r_i'$ and $|H^{p^i}| = |G^{p^i}|$, we have

$$\alpha_e = (r_e' - 1)(|G^{p^{e-1}}| - 1), \qquad \alpha_i = (r_i' - 1)(|G^{p^{i-1}}| - |G^{p^i}|) - (r_{i+1}' - 1)(|G^{p^i}| - |G^{p^{i+1}}|), \quad l < i < e.$$

For $i = l$, obviously $r_l' - 1 = r_l$ and $|G^{p^{l-1}}| = |H^{p^l}|p^{r_l+1}$. By Theorem 1.1 and the induction hypothesis,

$$\begin{aligned}\alpha_l + \beta_l &= (r_l - 1)(|H^{p^{l-1}}| - |H^{p^l}|) - (r_{l+1} - 1)(|H^{p^l}| - |H^{p^{l+1}}|) + |H^{p^l}|(r_lp^{r_l+1} - (r_l - 1)p^{r_l} - 1)\\ &= |H^{p^l}|(r_l - 1)(p^{r_l} - 1) - (r_{l+1}' - 1)(|G^{p^l}| - |G^{p^{l+1}}|) + |H^{p^l}|(r_lp^{r_l+1} - (r_l - 1)p^{r_l} - 1)\\ &= |H^{p^l}|r_l(p^{r_l+1} - 1) - (r_{l+1}' - 1)(|G^{p^l}| - |G^{p^{l+1}}|)\\ &= (r_l' - 1)(|G^{p^{l-1}}| - |G^{p^l}|) - (r_{l+1}' - 1)(|G^{p^l}| - |G^{p^{l+1}}|).\end{aligned}$$

For $1 \leq i \leq l$, all the r_i are equal, as are the r_i'. Let $r' = r_i'$ and $r = r_i$; then $r' - 1 = r$. Suppose $1 \leq i < l$; by the induction hypothesis,

$$\begin{aligned}\alpha_i &= (r - 1)(|H^{p^{i-1}}| - |H^{p^i}|) - (r - 1)(|H^{p^i}| - |H^{p^{i+1}}|)\\ &= (r - 1)|H^{p^{i+1}}|(p^r - 1)^2.\end{aligned}$$

By Theorem 1.1 and the fact $|H^{p^{i+1}}| = |H^{p^i}|p^{-r}$, we have

$$\begin{aligned}\beta_i &= |H^{p^i}|[rp^{-r}p^{l-i-1}(p^{r+1} - 1)^2 - (r - 1)p^{-r}(p^r - 1)^2]\\ &= |H^{p^{i+1}}|[rp^{l-i-1}(p^{r+1} - 1)^2 - (r - 1)(p^r - 1)^2].\\ &= r|H^{p^{i+1}}|p^{l-i-1}(p^{r+1} - 1)^2 - \alpha_i.\end{aligned}$$

Since

$$\begin{aligned}(r' - 1)(|G^{p^{i-1}}| - |G^{p^i}|) - (r' - 1)(|G^{p^i}| - |G^{p^{i+1}}|) &= r(|H^{p^{i-1}}|p^{l-i+1} - |H^{p^i}|p^{l-i}) - r(|H^{p^i}|p^{l-i} - |H^{p^{i+1}}|p^{l-i-1})\\ &= r|H^{p^{i+1}}|(p^{2r+l-i+1} - p^{r+l-i}) - r|H^{p^{i+1}}|(p^{r+l-i} - p^{l-i-1})\\ &= r|H^{p^{i+1}}|p^{l-i-1}(p^{r+1} - 1)^2,\end{aligned}$$

we have

$$\alpha_i + \beta_i = (r_i' - 1)(|G^{p^{i-1}}| - |G^{p^i}|) - (r_{i+1}' - 1)(|G^{p^i}| - |G^{p^{i+1}}|), \quad 1 \leq i < l.$$

Now we have proved that the formula for $K_2(\mathbb{F}_pG)$ is correct when $e > l$.

Next we consider the case $e = l$, that is, G is a homogeneous abelian p-group. Let r' and r denote the p^i-ranks of G and H respectively; for $i \leq e$, $r' - 1 = r$. By the induction hypothesis and Theorem 1.1,

$$\alpha_e = (r - 1)(|H^{p^{e-1}}| - 1) = (r - 1)(p^r - 1), \qquad \beta_e = rp^{r+1} - (r - 1)p^r - 1.$$

So

$$\alpha_e + \beta_e = r(p^{r+1} - 1) = (r' - 1)(|G^{p^{e-1}}| - 1).$$

If $1 \le i < l$,

$$\begin{aligned}
\alpha_i + \beta_i &= (r-1)(|H^{p^{i-1}}| - 2|H^{p^i}| + |H^{p^{i+1}}|) + |H^{p^i}|(rp^{e-r-i-1}(p^{r+1}-1)^2 - (r-1)p^{-r}(p^r-1)^2)\\
&= (r-1)|H^{p^{i+1}}|(p^{2r} - 2p^r + 1) + |H^{p^{i+1}}|(rp^{e-i-1}(p^{r+1}-1)^2 - (r-1)(p^r-1)^2)\\
&= r|H^{p^{i+1}}|p^{e-(i+1)}(p^{2r+2} - 2p^{r+1} + 1)\\
&= r|G^{p^{i+1}}|(p^{2r'} - 2p^{r'} + 1)\\
&= r(|G^{p^{i-1}}| - 2|G^{p^i}| + |G^{p^{i+1}}|).
\end{aligned}$$

So the formula for $K_2(\mathbb{F}_p G)$ holds if $e = l$. Now the theorem is proved. $\square$

Examples 3.6. Let us now determine the explicit structure of $K_2(\mathbb{F}_2[C_4 \times C_4])$. By (1.1) we know that $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2 C_4[t]/(t^4), (t))$. By Theorem 3.4, $K_2(\mathbb{F}_2 C_4[t]/(t^4), (t))$ is generated by Dennis–Stein symbols in the following six matrices, denoted by S.

$$\begin{aligned}
k = 1, &\quad \begin{pmatrix} \widehat{\langle t, t\rangle} & \langle \sigma t, t\rangle \\ \widehat{\langle \sigma^2 t, t\rangle} & \langle \sigma^3 t, t\rangle \end{pmatrix}, \quad \begin{pmatrix} \langle t, \sigma\rangle & \langle \sigma t, \sigma\rangle \\ \langle \sigma^2 t, \sigma\rangle & \langle \sigma^3 t, \sigma\rangle \end{pmatrix}.\\
k = 2, &\quad \begin{pmatrix} \widehat{\langle t^2, t\rangle} & \langle \sigma t^2, t\rangle \\ \widehat{\langle \sigma^2 t^2, t\rangle} & \langle \sigma^3 t^2, t\rangle \end{pmatrix}, \quad \begin{pmatrix} \widehat{\langle t^2, \sigma\rangle} & \widehat{\langle \sigma t^2, \sigma\rangle} \\ \widehat{\langle \sigma^2 t^2, \sigma\rangle} & \widehat{\langle \sigma^3 t^2, \sigma\rangle} \end{pmatrix}.\\
k = 3, &\quad \begin{pmatrix} \widehat{\langle t^3, t\rangle} & \langle \sigma t^3, t\rangle \\ \widehat{\langle \sigma^2 t^3, t\rangle} & \langle \sigma^3 t^3, t\rangle \end{pmatrix}, \quad \begin{pmatrix} \widehat{\langle t^3, \sigma\rangle} & \langle \sigma t^3, \sigma\rangle \\ \widehat{\langle \sigma^2 t^3, \sigma\rangle} & \langle \sigma^3 t^3, \sigma\rangle \end{pmatrix}.
\end{aligned}$$

As in Lemma 3.5, let

$$\begin{aligned}
T_1 &= \{\langle t, t\rangle, \langle \sigma^2 t, t\rangle, \langle t^2, t\rangle, \langle \sigma^2 t^2, t\rangle, \langle t^3, t\rangle, \langle \sigma^2 t^3, t\rangle\};\\
T_2 &= \{\langle t^2, \sigma\rangle, \langle \sigma^2 t^2, \sigma\rangle, \langle t^3, \sigma\rangle, \langle \sigma^2 t^3, \sigma\rangle\};\\
T_3 &= \{\langle \sigma t^2, \sigma\rangle, \langle \sigma^3 t^2, \sigma\rangle\}.
\end{aligned}$$

Set $T = T_1 \cup T_2 \cup T_3$; then $K_2(\mathbb{F}_2 C_4[t]/(t^4), (t))$ is generated by symbols in $S \setminus T$, and the symbols in T are given a hat in the above matrices to distinguish them from the other symbols. Using the notation from the proof of Theorem 1.1,

$$V = \{0, 1, 2, 3\}.$$

If $k = 1$, $s = 0$, so the partition of V is $N_{1,2} = \{0, 1\}$, $N_{1,1} = \{2, 3\}$. If $k = 2, 3$, $s = 1$, so the partition of V is $N_{2,1} = N_{3,1} = \{0, 1, 2, 3\}$. As in the proof of Theorem 1.1, the symbols of order 2^2 in $S \setminus T$ are

$$\langle \sigma t, t\rangle, \qquad \langle t, \sigma\rangle, \qquad \langle \sigma t, \sigma\rangle.$$

The Dennis–Stein symbols of order 2 in $S \setminus T$ are

$$\begin{aligned}
&\langle \sigma^3 t, t\rangle - \langle \sigma t, t\rangle,\ \langle \sigma^2 t, \sigma\rangle - \langle t, \sigma\rangle,\ \langle \sigma^3 t, \sigma\rangle - \langle \sigma t, \sigma\rangle,\\
&\langle \sigma t^2, t\rangle,\ \langle \sigma^3 t^2, t\rangle,\ \langle \sigma t^3, t\rangle,\ \langle \sigma^3 t^3, t\rangle,\ \langle \sigma t^3, \sigma\rangle,\ \langle \sigma^3 t^3, \sigma\rangle,
\end{aligned}$$

so we have $K_2(\mathbb{F}_2[C_4 \times C_4]) \cong K_2(\mathbb{F}_2 C_4[t]/(t^4), (t)) = C_4^3 \oplus C_2^9$.

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